International Journal of Theoretical Physics, Vol. 34, No. 5, 1995

Unified Gravitational and Yang–Mills Fields

Dave Pandres, Jr.[1]

Received August 26, 1994

We unify the gravitational and Yang–Mills fields by extending the diffeomorphisms in ($N = 4 + n$)-dimensional space-time to a larger group, called the conservation group. This is the largest group of coordinate transformations under which conservation laws are covariant statements. We present two theories that are invariant under the conservation group. Both theories have field equations that imply the validity of Einstein's equations for general relativity with the stress-energy tensor of a non-Abelian Yang–Mills field (with massive quanta) and associated currents. Both provide a geometrical foundation for string theory and admit solutions that describe the direct product of a compact n-dimensional space and flat four-dimensional space-time. One of the theories requires that the cosmological constant shall vanish. The conservation group symmetry is so large that there is reason to believe the theories are finite or renormalizable.

1. INTRODUCTION

In his autobiographical notes, Einstein (1949) suggested that the construction of a unified field theory "would be most beautiful, if one were to succeed in expanding the group once more, analogous to the step which led from special relativity to general relativity." In four prior papers (Pandres, 1962, 1981, 1984a,b), we pursued Einstein's suggestion that the diffeomorphisms (the covariance group for general relativity) somehow be extended to a larger group. In this Introduction and in Section 2 we recall the developments from these prior papers that are needed in subsequent sections. In recalling these developments, we shall make minor changes in notation and terminology to enhance clarity. A diffeomorphism from coordinates x^α to $x^{\bar\alpha}$ satisfies the commutation condition $[\partial_\mu, \partial_\nu]x^{\bar\alpha} = 0$, where $[\partial_\mu, \partial_\nu] = \partial_\mu\partial_\nu - \partial_\nu\partial_\mu$, and ∂_μ denotes partial differentiation with respect to x^μ. (Partial differentiation is denoted also by a comma, e.g., $\partial_\mu x^{\bar\alpha} = x^{\bar\alpha}{}_{,\mu}$.) Initially (Pandres, 1962), we

[1]Department of Mathematics and Computer Science, North Georgia College, Dahlonega, Georgia 30597.

0020-7748/95/0500-0733$07.50/0

merely proposed that this commutation condition be discarded. Our proposal was motivated by an argument, recalled in Section 2.1 of the present paper, which is a generalization of the "elevator" argument that led Einstein from special relativity to general relativity. In Section 2.2 we consider in detail why commutation of partial derivatives is not to be taken for granted. Briefly, the reason is that partial derivatives are defined on a class of *functionals on paths* $F(p)$ that contains the ordinary functions $F(x)$ as a subclass. The new coordinates $x^{\bar{\alpha}}$ are such path-dependent functionals. Subsequently (Pandres, 1981, 1984a,b) we proposed, more specifically, that the condition $[\partial_\mu, \partial_\nu]x^{\bar{\alpha}} = 0$ be replaced by the weaker condition

$$x^\nu_{,\bar{\alpha}}[\partial_\mu, \partial_\nu]x^{\bar{\alpha}} = 0$$

(We employ the summation convention. Greek and lowercase Latin indices take the values 0, 1, 2, . . . , $N - 1$.) Transformations satisfying this weaker condition are called *conservative*, and the group of such transformations is called the *space-time conservation group*. This is the largest group of coordinate transformations under which conservation laws are covariant statements. It is a proper subgroup of the path-dependent coordinate transformations, and contains the diffeomorphisms as a proper subgroup.

The theories developed in the present paper are similar to Kaluza–Klein theory in that we work in $N = 4 + n$ dimensions. However, we use simpler variational principles, and our Yang–Mills field will emerge without the "zero-mode approximation" of Kaluza–Klein theories. Modern Kaluza–Klein theories (Appelquist *et al.*, 1987) are based upon the N-dimensional Hilbert variational principle $\delta \int R\sqrt{-g}\, d^Nx = 0$, where R is the Ricci scalar (i.e., scalar under diffeomorphisms) and g is the determinant of the space-time metric $g_{\mu\nu}$. The field equations that flow from Hilbert's principle are the N-dimensional analog of Einstein's equations for the free gravitational field. The Einstein equations are not covariant under the space-time conservation group, because R is not an invariant under this group. Therefore, $R\sqrt{-g}$, the integrand in Hilbert's principle, is not a scalar density of weight $+1$ under the space-time conservation group, nor does it differ from such a scalar density by a pure divergence. By contrast, the quantity $\sqrt{-g}$ is a scalar density of weight $+1$ under *all* path-dependent coordinate transformations, including those belonging to the space-time conservation group. We recall the prophetic remark by Dirac (1930) that "Further progress lies in the direction of making our equations invariant under wider and still wider transformations." Accordingly, we use the variational principle

$$\delta \int \sqrt{-g}\, d^Nx = 0$$

which may be thought of as a "principle of stationary volume." Schrödinger

(1950) recognized that this is the simplest general relativistic variational principle one can write down, but he noted that if one varies $g_{\mu\nu}$, one obtains the Euler–Lagrange equations $\sqrt{-g}\ g^{\mu\nu} = 0$, which cannot serve as field equations. This difficulty persists if one writes $g_{\mu\nu} = g_{ij}h^i{}_\mu h^j{}_\nu$, where $g_{ij} = \text{diag}(-1, 1, \ldots, 1)$, and varies $h^i{}_\mu$. One obtains the Euler–Lagrange equations $\sqrt{-g}\ h_i{}^\mu = 0$, which cannot serve as field equations. We shall see in Section 3.1 that a major advantage of introducing path-dependent functionals is this: If we write

$$h^i{}_\mu = x^i{}_{,\mu}$$

and vary the functionals x^i, we obtain the Euler–Lagrange equations

$$(\sqrt{-g}h_i{}^\mu)_{,\mu} = 0$$

which are covariant under the space-time conservation group, and also under the group of transformations from coordinates x^i to $x^{\bar{i}}$ that satisfy the conditions

$$x^j{}_{,\bar{m}}[\partial_i, \partial_j]x^{\bar{n}} = 0$$

and

$$g_{\bar{m}\bar{n}} = g_{ij}x^i{}_{,\bar{m}}x^j{}_{,\bar{n}}$$

where $g_{\bar{m}\bar{n}} = g_{ij} = \text{diag}(-1, 1, \ldots, 1)$. This group is called the *frame conservation group*. The direct product of the space-time conservation group and the frame conservation group will be referred to simply as the *conservation group*.

If the x^i were ordinary functions, rather than path-dependent functionals, then our Euler–Lagrange equations would be trivial identities. Since the x^i are path-dependent functionals, however, these Euler–Lagrange equations serve as field equations for a very promising physical theory. In Section 3.4 we present evidence that the gauge symmetry of Yang–Mills theory is merely an approximation to our (exact) frame conservation group symmetry. This leads in a natural way to a "total" covariant derivative that differs from the usual covariant derivative with respect to the Christoffel symbol $\Gamma^\alpha{}_{\mu\nu}$ when the quantity differentiated has any Latin indices. We use a stroke | to denote this total covariant derivative. The affine connection for Latin indices is the negative of $A^i{}_{j\nu}$, where $A_{ij\nu}$ is the totally antisymmetric part of the Ricci rotation coefficient $\gamma_{ij\nu}$ under the permutation group of degree three. Thus,

$$h^i{}_{\mu|\nu} = h^i{}_{\mu,\nu} - h^i{}_\alpha\Gamma^\alpha{}_{\mu\nu} - h^j{}_\mu A^i{}_{j\nu}$$

and $h_{i\mu|\nu} = h_{i\mu,\nu} - h_{i\alpha}\Gamma^\alpha{}_{\mu\nu} + h_{j\mu}A^j{}_{i\nu}$. The use of our total covariant derivative is equivalent to the use of a spin connection with torsion. (The values $i = 0, 1, 2, 3$ are related to electromagnetism and spin angular momentum, while the values $i = 4, 5, \ldots, N - 1$ are related to the short-range interactions and the isospin-type variables of modern gauge theory.) We define the Yang–Mills field $M_{\mu\nu i}$ as the mixed symmetry part of the Ricci rotation coefficient $\gamma_{\mu\nu i}$. With this definition, it turns out that $M_{\mu\nu}{}^{i}$ is just the "total curl" of $h^{i}{}_{\mu}$, i.e.,

$$M_{\mu\nu}{}^{i} = h^{i}{}_{\nu|\mu} - h^{i}{}_{\mu|\nu}$$

We define a total Riemann tensor by

$$\mathfrak{R}^{\alpha}{}_{\beta\mu\nu} = h_{i}{}^{\alpha}(h^{i}{}_{\beta|\mu|\nu} - h^{i}{}_{\beta|\nu|\mu})$$

and a total Einstein tensor by

$$\mathfrak{G}_{\mu\nu} = \mathfrak{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathfrak{R}$$

where $\mathfrak{R}_{\mu\nu} = \mathfrak{R}^{\alpha}{}_{\mu\alpha\nu}$, and $\mathfrak{R} = \mathfrak{R}^{\alpha}{}_{\alpha}$. In Section 4 we show that our field equations imply the validity of

$$\mathfrak{G}_{\underline{\mu\nu}} = \tfrac{1}{2}(J_{\mu i}h^{i}{}_{\nu} + J_{\nu i}h^{i}{}_{\mu}) - (M^{\alpha}{}_{\mu i}M_{\alpha\nu}{}^{i} - \tfrac{1}{4}g_{\mu\nu}M^{\alpha\beta i}M_{\alpha\beta i})$$

where $\mathfrak{G}_{\underline{\mu\nu}}$ is the symmetric part of $\mathfrak{G}_{\mu\nu}$ and $J_{\mu i} = M_{\mu}{}^{\alpha}{}_{i|\alpha}$ is the total Yang–Mills current. These are just total Einstein equations with the stress-energy tensor of a non-Abelian Yang–Mills field and associated currents, and with a vanishing cosmological constant. In Section 3 we present evidence that the quanta of the field $M_{\mu\nu}{}^{i}$ are massive.

In Section 5 we obtain a large class of solutions to our field equations. This class includes solutions that describe the direct product of a compact n-dimensional space and flat four-dimensional space-time. Failure to admit such direct product solutions for $n > 1$ is the main weakness of classical, purely bosonic Kaluza–Klein theory (Appelquist *et al.*, 1987).

In Section 6 we discuss an alternate theory that is also covariant under the conservation group. The variational principle for this alternate theory is

$$\delta \int (C^{\mu}C_{\mu} + \Lambda)\sqrt{-g}\, d^{N}x = 0$$

where $C_{\mu} = h_{i}{}^{\nu}(h^{i}{}_{\mu,\nu} - h^{i}{}_{\nu,\mu})$ and Λ is a constant. When one varies $h^{i}{}_{\mu}$, one obtains field equations that imply the validity of the total Einstein equations with the stress-energy tensor of a non-Abelian Yang–Mills field (with massive quanta) and associated currents, but with a cosmological constant that need not vanish. In Sections 6.1 and 6.2 we discuss the relative merits of the two theories.

Quantization of our theories will be discussed in a future paper. Here, we merely note that the conservation group symmetry is so large that there is reason to believe the theories are finite or renormalizable. Indeed, the symmetry is so large that points (and point particles) do not have invariant meaning. Points appear to have invariant meaning only if one fails to recognize the role played by the conservation group and hence limits one's transformations to the diffeomorphisms. However, paths (and path-particles, i.e., strings) have invariant meaning. Thus, our theories provide the much-sought (Witten, 1988) geometrical foundation for string theory.

2. MOTIVATION AND MATHEMATICAL PRELIMINARIES

2.1. Motivation

In our first paper on field unification (Pandres, 1962) we began with the special relativistic equation of motion for a free particle $d^2x^i/ds^2 = 0$, where $-ds^2 = g_{ij}dx^i dx^j$, and $g_{ij} = \mathrm{diag}(-1, 1, 1, 1)$. We considered the image equation of this free-particle equation under a transformation from coordinates x^i to coordinates x^α, where $[\partial_\mu, \partial_\nu]x^i \neq 0$. This image equation is

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -f^\alpha{}_\nu\frac{dx^\nu}{ds} \tag{1}$$

where $\Gamma^\alpha{}_{\mu\nu}$ is the Christoffel symbol computed from the metric $g_{\mu\nu} = g_{ij}x^i{}_{,\mu}x^j{}_{,\nu}$, and $f_{\mu\nu} = v_i f^i{}_{\mu\nu}$, where $f^i{}_{\mu\nu} = \partial_\mu x^i{}_{,\nu} - \partial_\nu x^i{}_{,\mu}$ and $v^i = dx^i/ds$ is the (constant) first integral of $d^2x^i/ds^2 = 0$. If $[\partial_\mu, \partial_\nu]x^i{}_{,\alpha} = 0$, then $f_{\mu\nu}$ satisfies $f_{\mu\nu,\alpha} + f_{\alpha\mu,\nu} + f_{\nu\alpha,\mu} = 0$, which are Maxwell's equations. Thus, it appears at first glance that Eq. (1) describes a charged particle moving in a gravitational and electromagnetic field. However, $f_{\mu\nu}$ cannot be interpreted as the electromagnetic field, because the relation $v^i = dx^i/ds = x^i{}_{,\mu}dx^\mu/ds$ implies that v^i depends upon dx^μ/ds. Although $f_{\mu\nu}$ is a linear combination of the $f^i{}_{\mu\nu}$ with coefficients v_i which are constant along the world line of a particle, it is unsatisfactory that the values of these coefficients should depend upon dx^μ/ds. This would imply that the electromagnetic field experienced by a particle depends upon the velocity of the particle, in disagreement with experiment. In two later papers (Pandres, 1973, 1977) we tried to get a satisfactory description of gravitation and electromagnetism alone by investigating a class of theories in which the antisymmetric tensors $f^i{}_{\mu\nu}$ are constant multiples of one another. This is the case if the transformation from x^i to x^α has the property that the four quantities $[\partial_\mu, \partial_\nu]x^i$ are constant multiples of one another. However, transformations with this property do not form a group. Therefore, theories in this class suffer from the defect that they are not motivated by group-theoretic arguments, and therefore lack a "guiding principle" such as the principle of general covariance. The inescapable fact is that for transformation groups with $[\partial_\mu, \partial_\nu]x^i \neq 0$, the right side of equation (1) contains four linearly independent antisymmetric tensors, while only one is needed to describe the electromagnetic field. This raises the intriguing question whether the antisymmetric tensors $f^i{}_{\mu\nu}$ might somehow, collectively, describe the Yang–Mills fields of modern gauge theory. We shall see that this is indeed the case.

2.2. Mathematical Preliminaries

Any ordered set of N independent real variables x^α may be regarded as coordinates of points in an N-dimensional arithmetic space X.

1. Paths. Let $x^\alpha(\lambda)$ be continuous functions of a real parameter λ on the interval $-\infty < \lambda < \infty$. By a *path p*, we mean the set of all points in X that are identified by $x^\alpha = x^\alpha(\lambda)$ for $-\infty < \lambda \leq \Lambda$. Thus, one endpoint of a path p is the point i with coordinates $\lim_{\lambda\to-\infty} x^\alpha(\lambda)$, while the other endpoint is the point x with coordinates $x^\alpha(\Lambda)$. We regard i as the initial point, and x as the terminus, of p. The set of all paths p is regarded as a space of paths and is denoted by P.

2. Path-Dependent Functionals and Their Derivatives. Let F be a path-dependent functional, i.e., a rule that assigns to each path p a real number $F(p)$. Following the method introduced by us (Pandres, 1962) and independently by Mandelstam (1962), we define derivatives of $F(p)$ by giving p an extension from its terminus x, while holding the rest of p completely fixed. Any path may be extended in this way by extending the domain of $x^\alpha(\lambda)$ to the interval $-\infty < \lambda \leq \Lambda + \Delta\Lambda$, where $\Delta\Lambda > 0$. The resulting path $p + \Delta p$ is called a path extended from p, and the set of all points in X that are defined by $x^\alpha = x^\alpha(\lambda)$ for $\Lambda < \lambda \leq \Lambda + \Delta\Lambda$ is called an extension of p and is denoted by Δp. If, for each path p and each extension Δp, the condition

$$\lim_{\Delta\Lambda\to 0}[F(p + \Delta p) - F(p)] = 0$$

is satisfied, we call F a *normal* functional. We limit our considerations to normal functionals. We define F' by

$$F' = \lim_{\Delta\Lambda\to 0} \frac{F(p + \Delta p) - F(p)}{\Delta\Lambda}$$

If the extension Δp is chosen so that, along it, only a single coordinate x^β changes, and if the parametrization is such that on this extension $\Delta\Lambda = \Delta x^\beta$, then F' is called the partial derivative of F with respect to x^β, and denoted by $\partial_\beta F$ or by $F_{,\beta}$. If, along Δp, the coordinate increments Δx^β are unrestricted and independent, then F' is called the total derivative of F with respect to Λ, and is denoted by $dF/d\Lambda$. It is also convenient to denote $dx^\alpha/d\lambda$, evaluated for $\lambda = \Lambda$, by $dx^\alpha/d\Lambda$. If the partial derivatives and the total derivative of F are related in such a way that the chain rule for differentiation is valid, i.e., if $dF/d\Lambda = F_{,\alpha}\, dx^{\alpha}/d\Lambda$, then F is called a *smooth* functional. A smooth functional whose partial derivatives of all orders are also smooth is called a *regular* functional. We limit our considerations to regular functionals. When we wish to emphasize the path-dependent character of a functional F, we use the notation $F(p)$. Our functionals include, as a subclass, the ordinary functions of x, i.e., functionals which are "path-dependent" in the trivial sense that they depend only on the terminus x of a path p; for them, we use the notation $F(x)$.

3. Noncommutativity of Partial Derivatives. From p, let two extended paths $p + \Delta p_1$ and $p + \Delta p_2$ be constructed such that the extensions Δp_1 and Δp_2 do not completely coincide, but such that the termini of $p + \Delta p_1$ and $p + \Delta p_2$ *do* coincide. The values of $F(p + \Delta p_1)$ and $F(p + \Delta p_2)$ are not generally equal. By letting Δp_1 be an extension along which first only x^{ν} changes and then only x^{μ} changes, and letting Δp_2 be an extension along which first only x^{μ} changes and then only x^{ν} changes, we see that $\partial_{\mu}\partial_{\nu}F$ equals $\partial_{\nu}\partial_{\mu}F$ for functions $F(x)$, but not generally for functionals $F(p)$.

4. The Requirement That No "Preferred" Coordinate System Shall Exist. From the chain rule, we have $F_{,\nu} = F_{,\tilde{\sigma}}x^{\tilde{\sigma}}{}_{,\nu}$. If we differentiate with respect to x^{μ} and subtract the corresponding expression with μ and ν interchanged, we get

$$[\partial_{\mu}, \partial_{\nu}]F = x^{\tilde{\rho}}{}_{,\mu}x^{\tilde{\sigma}}{}_{,\nu}[\partial_{\tilde{\rho}}, \partial_{\tilde{\sigma}}]F + F_{,\tilde{\sigma}}[\partial_{\mu}, \partial_{\nu}]x^{\tilde{\sigma}}$$

For $[\partial_{\mu}, \partial_{\nu}]x^{\tilde{\sigma}} \neq 0$, if we were to demand that $[\partial_{\mu}, \partial_{\nu}]F$ vanish, then we would find that $[\partial_{\tilde{\rho}}, \partial_{\tilde{\sigma}}]F$ does not generally vanish. Thus, the coordinates x^{α} and $x^{\tilde{\alpha}}$ would not be on an equal footing; i.e., the coordinates x^{α} would be "preferred." The requirement that x^{α} and $x^{\tilde{\alpha}}$ be on an equal footing compels us to consider a space in which paths, rather than points, are the primary elements.

5. Abstract Path Space. Just as the x^{α} are regarded as coordinates of points x in the arithmetic space X and the set of all paths p is regarded as a space of paths P, *another* ordered set of N independent real variables $x^{\tilde{\alpha}}$ may be regarded as coordinates of points $\tilde{x}$ in *another* N-dimensional arithmetic space $\tilde{X}$, and the set of all paths $\tilde{p}$ may be regarded as *another* space of paths $\tilde{P}$. Let M be a one-to-one mapping from P onto $\tilde{P}$; let $\tilde{p}$ be the image path of p, and let $\tilde{x}$ be the terminus of $\tilde{p}$. Since $\tilde{x}$ is determined by $\tilde{p}$, and $\tilde{p}$ is determined by p (via the mapping M), it is clear that the coordinates $x^{\tilde{\alpha}}$ are functionals of p, i.e., $x^{\tilde{\alpha}} = x^{\tilde{\alpha}}(p)$. Similarly, $x^{\alpha} = x^{\alpha}(\tilde{p})$. If the image path of each path extended from p is a path extended from $\tilde{p}$, and if $x^{\tilde{\alpha}}(p)$ and $x^{\alpha}(\tilde{p})$ are regular functionals, then M is called a regular mapping. We limit our considerations to regular mappings.

We began by regarding a mapping M as a *path transformation* (which maps each path p in P to a path $\tilde{p}$ in $\tilde{P}$ and conversely). There is, however, another point of view that is more interesting and useful, and that we now adopt: We introduce an *abstract path space* $\mathfrak{P}$ in which (abstract) paths $\mathfrak{p}$ are the primary elements, and regard M as a path-dependent coordinate transformation $x^{\tilde{\alpha}} = x^{\tilde{\alpha}}(p)$ that merely changes the arithmetic-space framework for discussing $\mathfrak{P}$. The arithmetic spaces X and $\tilde{X}$ provide equivalent frameworks for discussing $\mathfrak{P}$, and the path spaces P and $\tilde{P}$ are equivalent representations of $\mathfrak{P}$. A path p and its image path $\tilde{p}$ are equivalent representations of the same abstract path $\mathfrak{p}$ in $\mathfrak{P}$. The changed point of view that we have adopted is analogous to that in which one begins by regarding a suitable transformation $x^{\tilde{\alpha}} = x^{\tilde{\alpha}}(x)$ as a mapping from a point x to a point $\tilde{x}$ and then recognizes that it is more interesting and useful to regard the transformation as a diffeomorphism, in which the same point of an abstract point space (a manifold) is merely relabeled with new coordinate values.

Many investigators, beginning with Eddington (1924) and continuing to the present (see, e.g., Davies and Brown, 1988), have expressed skepticism that a manifold adequately describes physical space. We propose that physical space be described by an abstract path space $\mathfrak{P}$. (We note that the path-integral formulation of quantum theory suggests that path space is the natural setting for a complete geometrical treatment of physics.) Path space possesses properties which are sufficiently close to what one conceives of intuitively as a space so that one may use it almost exactly as one conventionally uses a manifold. The main mathematical difference is that partial derivatives do not generally commute, so when one has repeated partial derivatives, one must carefully preserve their orders.

The coordinates x^{α} and $x^{\tilde{\alpha}}$ provide equivalent coordinate systems for discussing $\mathfrak{P}$, but the points x and $\tilde{x}$ that x^{α} and $x^{\tilde{\alpha}}$ identify in X and $\tilde{X}$, respectively, have no meaning in $\mathfrak{P}$. This is clear, because a path-dependent coordinate transformation does not generally establish a one-to-one correspondence between points of X and $\tilde{X}$, even in coordinate patches. The correspondence between x and $\tilde{x}$ is both one-to-many and many-to-one (hence, nonunique in both directions). Thus, an assertion that a particle is located at a particular point has no (invariant) meaning. However, an assertion that a particle is distributed along a particular path has meaning. Accordingly, a serendipitous result of our theories is that they provide a geometrical foundation for string theory. Witten (1988), especially, has emphasized the need for such a foundation. Many paths in P with the same termini may have image paths in $\tilde{P}$ with different termini, and conversely. Thus, an assertion that an abstract path, or a string, is closed (or is not closed) has no invariant meaning. Such an assertion appears to have meaning only if one fails to recognize the role played by the path-dependent coordinate transformations and, in the belief that the path space is a manifold, restricts one's transformations to the diffeomorphisms. How could such a belief arise? The answer is this: There exists a class of solutions to our field equations for which the x^i are "nonintegrable functions" of the type advocated by Dirac (1978). His nonintegrable functions are functionals $F(p)$ whose derivatives are functions $F_{,\mu}(x)$. (In Section 5 we exhibit a large set of solutions belonging to this class.) For these solutions, the $h^i{}_\mu$ defined by $h^i{}_\mu = x^i{}_{,\mu}$ are ordinary functions; hence $[\partial_\alpha, \partial_\beta]h^i{}_\mu = 0$. If one believes that the fundamental fields are these $h^i{}_\mu$ or other fields constructed from them, then one can work *exactly* as if the path space was a manifold. Thus, one could easily be deluded into believing that the path space is a manifold. Such a belief, however, would have profound physical consequences. Perhaps the most important consequence, is this: An observer who lives in a path space, but believes that he lives in a curved Riemannian manifold, would "see" effects which he would interpret as resulting from the presence of a Yang–Mills gauge field. This is analogous to the familiar case in which an observer who lives in a curved Riemannian manifold, but believes that he lives in a flat manifold, "sees" effects which he interprets as resulting from the presence of a gravitational field.

6. The Space-Time Conservation Group. A relativistic conservation law is an expression of the form $V^\alpha{}_{,\alpha} = 0$ where V^α is a vector density of weight $+1$. This is a covariant statement under a path-dependent coordinate transformation relating x^α and $x^{\tilde\alpha}$ if and only if it implies and is implied by the relation $V^{\tilde\alpha}{}_{,\tilde\alpha} = 0$. The transformation law for a vector density of weight $+1$ is

$$V^{\tilde\alpha} = \frac{\partial x}{\partial \tilde x} x^{\tilde\alpha}{}_{,\mu} V^\mu \tag{2}$$

where $\partial x/\partial \tilde x$ is the (nonzero) determinant of $x^\mu{}_{,\tilde\alpha}$. Upon differentiating Eq. (2) with respect to $x^{\tilde\alpha}$, we obtain

$$V^{\tilde\alpha}{}_{,\tilde\alpha} = \left(\frac{\partial x}{\partial \tilde x} x^{\tilde\alpha}{}_{,\mu}\right)_{,\tilde\alpha} V^\mu + \frac{\partial x}{\partial \tilde x} V^\mu{}_{,\mu}$$

For arbitrary V^μ we see that a conservation law is a covariant statement if and only if

$$\left(\frac{\partial x}{\partial \tilde x} x^{\tilde\alpha}{}_{,\mu}\right)_{,\tilde\alpha} = 0 \tag{3}$$

For this reason, we call a path-dependent coordinate transformation *conservative* if it satisfies equation (3). Now,

$$\left(\frac{\partial x}{\partial \tilde x} x^{\tilde\alpha}{}_{,\mu}\right)_{,\tilde\alpha} = \left(\frac{\partial x}{\partial \tilde x}\right)_{,\tilde\alpha} x^{\tilde\alpha}{}_{,\mu} + \frac{\partial x}{\partial x} x^{\tilde\alpha}{}_{,\mu,\tilde\alpha} = \partial_\mu \frac{\partial x}{\partial \tilde x} + \frac{\partial x}{\partial \tilde x} x^{\tilde\alpha}{}_{,\mu,\nu} x^\nu{}_{,\tilde\alpha}$$

so, if we use the well-known formula

$$\partial_\mu \frac{\partial x}{\partial \tilde{x}} = \frac{\partial x}{\partial \tilde{x}} x^{\tilde{\alpha}}{}_{,\nu} \partial_\mu x^\nu{}_{,\tilde{\alpha}}$$

for the derivative of a determinant, and note that $x^{\tilde{\alpha}}{}_{,\nu} x^\nu{}_{,\tilde{\alpha},\mu} = -x^{\tilde{\alpha}}{}_{,\nu,\mu} x^\nu{}_{,\tilde{\alpha}}$, we find that equation (3) may be expressed in the equivalent form

$$x^\nu{}_{,\tilde{\alpha}}[\partial_\mu, \partial_\nu]x^{\tilde{\alpha}} = 0 \tag{4}$$

We note that equation (4) is satisfied if $[\partial_\mu, \partial_\nu]x^{\tilde{\alpha}} = 0$. Thus, we see that each diffeomorphism is a conservative coordinate transformation, but that the converse is not true. We now recall (Pandres, 1981) an explicit proof that the conservative coordinate transformations form a group. [Finkelstein (1981), however, pointed out that the group property follows implicitly from the derivation given above.] First, we note that the identity transformation $x^{\tilde{\alpha}} = x^\alpha$ is a conservative coordinate transformation. Next, we consider the result of following a coordinate transformation from x^α to $x^{\tilde{\alpha}}$ by a coordinate transformation from $x^{\tilde{\alpha}}$ to $x^{\hat{\alpha}}$. Upon differentiating

$$x^{\hat{\alpha}}{}_{,\mu} = x^{\hat{\alpha}}{}_{,\tilde{\rho}} x^{\tilde{\rho}}{}_{,\mu} \tag{5}$$

with respect to x^ν, subtracting the corresponding expression with μ and ν interchanged, and multiplying by $x^\nu{}_{,\hat{\alpha}}$ we obtain

$$x^\nu{}_{,\hat{\alpha}}[\partial_\mu, \partial_\nu]x^{\hat{\alpha}} = x^{\tilde{\rho}}{}_{,\mu} x^{\tilde{\sigma}}{}_{,\hat{\alpha}}[\partial_{\tilde{\rho}}, \partial_{\tilde{\sigma}}]x^{\hat{\alpha}} + x^\nu{}_{,\tilde{\rho}}[\partial_\mu, \partial_\nu]x^{\tilde{\rho}} \tag{6}$$

We see from equation (6) that if $x^\nu{}_{,\tilde{\rho}}[\partial_\mu, \partial_\nu]x^{\tilde{\rho}}$ and $x^{\tilde{\sigma}}{}_{,\hat{\alpha}}[\partial_{\tilde{\rho}}, \partial_{\tilde{\sigma}}]x^{\hat{\alpha}}$ vanish, then $x^\nu{}_{,\hat{\alpha}}[\partial_\mu, \partial_\nu]x^{\hat{\alpha}}$ vanishes. This shows that if the transformations from x^α to $x^{\tilde{\alpha}}$ and from $x^{\tilde{\alpha}}$ to $x^{\hat{\alpha}}$ are conservative coordinate transformations, then the product transformation from x^α to $x^{\hat{\alpha}}$ is a conservative coordinate transformation. If we let $x^{\hat{\alpha}} = x^\alpha$, we see from equation (6) that the inverse of a conservative coordinate transformation is a conservative coordinate transformation. From equation (5), we see that the product of matrices $x^{\tilde{\rho}}{}_{,\mu}$ and $x^{\hat{\alpha}}{}_{,\tilde{\rho}}$ (which represent the transformations from x^α to $x^{\tilde{\alpha}}$ and from $x^{\tilde{\alpha}}$ to $x^{\hat{\alpha}}$, respectively) equals the matrix $x^{\hat{\alpha}}{}_{,\mu}$ (which represents the product transformation from x^α to $x^{\hat{\alpha}}$). It is obvious, and well known, that if products admit a matrix representation in this sense, then the associative law is satisfied. This completes the proof that the conservative coordinate transformations form a group, which we call the space-time conservation group.

We note that if $[\partial_\mu, \partial_\nu]x^{\tilde{\alpha}} = 0$, then equation (4) is satisfied, i.e., the space-time conservation group contains the diffeomorphisms as a subgroup. Thus, to show that it contains the diffeomorphisms as a *proper* subgroup, we need only exhibit a coordinate transformation that satisfies equation (4), but does not satisfy $[\partial_\mu, \partial_\nu]x^{\tilde{\alpha}} = 0$. Such a coordinate transformation is

$$x^{\bar\alpha} = x^{\alpha} + \delta_0^{\alpha} \int_i^x x^1\, dx^2 \tag{7}$$

where δ_μ^α is the usual Kronecker delta, and the integral from i to x is taken along the path p. We see by inspection that the inverse of the transformation defined in equation (7) is

$$x^{\nu} = x^{\bar\nu} - \delta_{\bar 0}^{\nu} \int_{\bar i}^{\bar x} x^{\bar 1} dx^{\bar 2} \tag{8}$$

where the integral from $\bar i$ to $\bar x$ is taken along the path $\bar p$. Upon differentiating equation (7) with respect to x^ν, we obtain $x^{\bar\alpha}{}_{,\nu} = \delta_\nu^\alpha + \delta_0^\alpha \delta_\nu^2 x^1$. By differentiating this with respect to x^μ and subtracting the corresponding expression with μ and ν interchanged, we obtain

$$[\partial_\mu, \partial_\nu] x^{\bar\alpha} = \delta_0^\alpha(\delta_\mu^1 \delta_\nu^2 - \delta_\nu^1 \delta_\mu^2) \tag{9}$$

A nonzero component of equation (9) is $[\partial_1, \partial_2] x^{\bar 0} = 1$, which shows that the coordinate transformation defined in equation (7) is not a diffeomorphism. Upon differentiating equation (8) with respect to $x^{\bar\alpha}$, we obtain $x^{\nu}{}_{,\bar\alpha} = \delta_\alpha^\nu - \delta_{\bar 0}^\nu \delta_{\bar\alpha}^{\bar 2} x^{\bar 1}$. Upon multiplying this and equation (9), we see that equation (4) is satisfied.

7. The Frame Conservation Group. Any ordered set of N independent real variables x^i may be regarded as coordinates of points in a N-dimensional "Latin" arithmetic space (just as the x^α are regarded as coordinates of points in the "Greek" arithmetic space X). Such a Latin space will be called a *frame*. Under a transformation between frames defined by the Latin coordinates x^i and $x^{\bar i}$, the transformation law for $h^i{}_\mu$ is $h^i{}_\mu = x^i{}_{,\bar m} h^{\bar m}{}_\mu$. The requirement that x^i and $x^{\bar i}$ be on an equal footing implies that we must allow only transformations which preserve the "Latin metric" g_{ij}, i.e., transformations which satisfy the condition

$$g_{\bar m \bar n} = g_{ij} x^i{}_{,\bar m} x^j{}_{,\bar n} \tag{10}$$

where $g_{\bar m \bar n} = g_{ij} = \mathrm{diag}(-1, 1, \ldots, 1)$. A transformation that satisfies equation (10) will be called a *frame transformation*, and the group of such transformations will be called the *frame transformation group*. The frame transformation group is $O(1, N - 1)$, which is a p-parameter Lie group, where $p = \frac{1}{2}N(N - 1)$. Thus $x^i{}_{,\bar m}$ depends upon parameters $\theta^1, \ldots, \theta^p$. If the θ's are constant, a frame transformation is called *global*. If the θ's are functions $\theta(x)$, the frame transformation is called *local*. A frame transformation that satisfies the condition

$$\left(\frac{\partial \chi}{\partial \bar\chi} x^{\bar m}{}_{,i}\right)_{,\bar m} = 0 \tag{11}$$

where $\partial\chi/\partial\bar{\chi}$ is the determinant of $x^i{}_{,\bar{m}}$, is called *conservative*, and the group of such transformations is called the *frame conservation group*. We note that equation (11) may be written

$$x^j{}_{,\bar{m}}[\partial_i, \partial_j]x^{\bar{m}} = 0 \tag{12}$$

via an argument analogous to that which leads from equation (3) to equation (4). We see from equation (10) that $\partial\chi/\partial\bar{\chi} = \pm 1$. Therefore, equation (11) may be written in the form

$$x^{\bar{m}}{}_{,i,\bar{m}} = 0 \tag{13}$$

The only frame transformations that establish a one-to-one correspondence between x^i and $x^{\bar{i}}$ are the global ones. Thus, only the global frame transformations are "frame diffeomorphisms." However, there exist local frame transformations that are conservative. Green (1991) noted that an example of a conservative frame transformation is

$$x^{\bar{m}} = \delta_0^m x^0 + \delta_3^m x^3 + \int_i^x [(\delta_1^m \cos x^3 + \delta_2^m \sin x^3)dx^1 + (\delta_2^m \cos x^3 - \delta_1^m \sin x^3)dx^2] \tag{14}$$

From equation (14), we find that

$$x^{\bar{m}}{}_{,i} = \delta_0^m\delta_i^0 + \delta_3^m\delta_i^3 + (\delta_1^m\delta_i^1 + \delta_2^m\delta_i^2)\cos x^3 + (\delta_2^m\delta_i^1 - \delta_1^m\delta_i^2)\sin x^3 \tag{15}$$

i.e., that

$$x^{\bar{m}}{}_{,i} = \begin{pmatrix} x^{\bar{0}}{}_{,0} & x^{\bar{0}}{}_{,1} & x^{\bar{0}}{}_{,2} & x^{\bar{0}}{}_{,3} \\ x^{\bar{1}}{}_{,0} & x^{\bar{1}}{}_{,1} & x^{\bar{1}}{}_{,2} & x^{\bar{1}}{}_{,3} \\ x^{\bar{2}}{}_{,0} & x^{\bar{2}}{}_{,1} & x^{\bar{2}}{}_{,2} & x^{\bar{2}}{}_{,3} \\ x^{\bar{3}}{}_{,0} & x^{\bar{3}}{}_{,1} & x^{\bar{3}}{}_{,2} & x^{\bar{3}}{}_{,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos x^3 & -\sin x^3 & 0 \\ 0 & \sin x^3 & \cos x^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

It is obvious from the last matrix in equation (16) that equation (14) defines a frame transformation. It is just a local rotation in the 1–2 plane. Since $x^{\bar{m}}{}_{,i}$ depends only upon x^3, and since $x^3 = x^{\bar{3}}$, we see that

$$x^{\bar{m}}{}_{,i,\bar{m}} = x^{\bar{3}}{}_{,i,\bar{3}} = x^{\bar{3}}{}_{,i,3}$$

Then, since $x^{\bar{3}}{}_{,i} = \text{const}$, we find that $x^{\bar{3}}{}_{,i,3} = 0$. Thus, we see that (13) is satisfied, i.e., that the frame transformation defined by (14) is conservative.

3. FIELD EQUATIONS

As we stated in Section 1, Schrödinger (1950) considered the simplest general relativistic variational principle one can write down, i.e., the "principle of stationary volume"

$$\delta \int \sqrt{-g}\, d^N x = 0 \tag{17}$$

3.1. Euler–Lagrange Equations

Since $\delta(-g)^{1/2} = -\frac{1}{2}(-g)^{-1/2}\delta g = -\frac{1}{2}(-g)^{-1/2} g g^{\mu\nu}\delta g_{\mu\nu}$, equation (17) may be written

$$\int \sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}\, d^N x = 0 \tag{18}$$

From equation (18), we see that if we vary $g_{\mu\nu}$, we obtain the Euler–Lagrange equations $\sqrt{-g}\, g^{\mu\nu} = 0$, which cannot serve as field equations for a physical theory (as Schrödinger noted). This difficulty persists if we write

$$g_{\mu\nu} = g_{ij} h^i{}_\mu h^j{}_\nu \tag{19}$$

where $g_{ij} = g^{ij} = \mathrm{diag}(-1, 1, \ldots, 1)$. Latin indices are raised and lowered by using g^{ij} and g_{ij} just as Greek indices are raised and lowered by using $g^{\mu\nu}$ and $g_{\mu\nu}$. We note that

$$g^{\mu\nu}\delta g_{\mu\nu} = g^{\mu\nu} g_{ij}(h^i{}_\mu \delta h^j{}_\nu + h^j{}_\nu \delta h^i{}_\mu) = 2g^{\mu\nu} g_{ij} h^j{}_\nu \delta h^i{}_\mu = 2h_i{}^\mu \delta h^i{}_\mu$$

Thus, equation (18) may be written

$$\int \sqrt{-g}\, h_i{}^\mu \delta h^i{}_\mu\, d^N x = 0 \tag{20}$$

From equation (20), we see that if we vary $h^i{}_\mu$, we get the Euler–Lagrange equations $\sqrt{-g}\, h_i{}^\mu = 0$, which cannot serve as field equations for a physical theory. If, however, we represent $h^i{}_\mu$ as the derivative of functionals $x^i(p)$, i.e.,

$$h^i{}_\mu = x^i{}_{,\mu} \tag{21}$$

and vary x^i, we find that equation (20) may be written $\int \sqrt{-g}\, h_i{}^\mu (\delta x^i)_{,\mu}\, d^N x = 0$. Integration by parts gives

$$\int (\sqrt{-g}\, h_i{}^\mu\, \delta x^i)_{,\mu}\, d^N x - \int (\sqrt{-g}\, h_i{}^\mu)_{,\mu} \delta x^i\, d^N x = 0$$

By using Gauss' theorem, we may write $\int (\sqrt{-g}\, h_i{}^\mu \delta x^i)_{,\mu}\, d^N x$ as an integral over the boundary of the region of integration in the arithmetic space X. We discard this boundary integral by demanding that δx^i shall vanish on the boundary. Thus, we obtain $\int (\sqrt{-g}\, h_i{}^{\mu})_{,\mu} \delta x^i \, d^N x = 0$, and by demanding that δx^i be arbitrary in the interior of the (arbitrary) region of integration in X, we obtain the Euler–Lagrange equations

$$(\sqrt{-g}\, h_i{}^{\mu})_{,\mu} = 0 \tag{22}$$

which are our field equations

Digression: Why the Integration Is Not over Path Space. Since our physical space is a space of paths, it may seem that our variational principle should be written as an integral over path space. This is not possible, however, because we do not have neighborhoods in path space. [Previously (Pandres, 1981, 1984a), we attempted to impose a topology on path space by defining a "distance" between two arbitrary paths p_1 and p_2. However, such a distance is not invariant under path-dependent coordinate transformations, so neighborhoods are not preserved under path-dependent coordinate transformations.] The closest thing we have to neighborhoods are what we might call "extension neighborhoods." The extension neighborhood of a path p is that path together with all paths $p + \Delta p$ which are extended from p. Integration over the extension neighborhood of a path p is mathematically equivalent to integration over a neighborhood (in the usual sense) in the arithmetic space X.

3.2. Alternate Forms of the Field Equations

We see from equation (19) that $\sqrt{-g}$ equals the determinant of $h^i{}_{\mu}$, which we denote by h. Now,

$$\begin{aligned}
0 &= (hh_j{}^{\alpha})_{,\alpha} \\
&= hh_{j,\alpha}^{\alpha} + h_j{}^{\alpha} h_{,\alpha} \\
&= hh_j{}^{\alpha}{}_{,\nu} \delta_{\alpha}^{\nu} + h_j{}^{\alpha} h h_i{}^{\nu} h^i{}_{\nu,\alpha} \\
&= hh_j{}^{\alpha}{}_{,\nu} h^i{}_{\alpha} h_i{}^{\nu} + hh_j{}^{\alpha} h_i{}^{\nu} h^i{}_{\nu,\alpha} \\
&= -hh_j{}^{\alpha} h^i{}_{\alpha,\nu} h_i{}^{\nu} + hh_j{}^{\alpha} h_i{}^{\nu} h^i{}_{\nu,\alpha} \\
&= hh_j{}^{\alpha} h_i{}^{\nu} (h^i{}_{\nu,\alpha} - h^i{}_{\alpha,\nu})
\end{aligned}$$

Upon multiplying this by $h^j{}_{\mu}$, we see that equation (22) may be written

$$h_i{}^{\nu}(h^i{}_{\nu,\mu} - h^i{}_{\mu,\nu}) = 0 \tag{23}$$

Since $h^i{}_{\mu} = x^i{}_{,\mu}$, equation (23) may be written

$$x^{\nu}{}_{,i}[\partial_{\mu}, \partial_{\nu}] x^i = 0 \tag{24}$$

This would be a trivial identity if the x^i were ordinary functions (for which

$[\partial_\mu, \partial_\nu]x^i = 0$), rather than path-dependent functionals. Comparison of equations (24) and (4) shows that our field equations just state that the transformation from x^i to x^μ is conservative, i.e., that path space is "conservatively flat."

It is important to recognize that conservative flatness does not imply flatness in the Riemannian sense. The analysis of Section 2.2.6 shows that

$$h^i{}_\mu = \delta^i{}_\mu + \delta^i_0 \delta^2_\mu x^1 \tag{25}$$

satisfies equation (23). However, we have shown (Pandres, 1981) that the metric $g_{\mu\nu} = g_{ij} h^i{}_\mu h^j{}_\nu$ formed from the $h^i{}_\mu$ of equation (25) yields a Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ which does not vanish. This is clear from an easy calculation which shows that it yields a Ricci scalar R with value 1/2. We also showed that the metric formed from the $h^i{}_\mu$ of equation (25) satisfies Einstein equations with the stress-energy tensor for an electrically charged dust cloud.

It is convenient to write equation (23) in the form

$$C_\mu = 0 \tag{26}$$

where

$$C_\mu = -h_i{}^\nu f^i{}_{\mu\nu} \tag{27}$$

and $f^i{}_{\mu\nu}$ is the curl of $h^i{}_\mu$, i.e.,

$$f^i{}_{\mu\nu} = \partial_\mu h^i{}_\nu - \partial_\nu h^i{}_\mu \tag{28}$$

We have previously (Pandres, 1981) shown that there exists a conservative space-time transformation from x^α to a special $x^{\bar\alpha}$ coordinate system in which $h^i{}_{\bar\mu}$ is constant if and only if C_μ vanishes. Thus, C_μ may be interpreted as a *curvature vector*, although this interpretation will not be needed in the present paper.

3.3. Covariance of the Field Equations

The quantity $\sqrt{-g}\, h_i{}^\mu$ is a vector density of weight $+1$ under transformations from x^α to $x^{\bar\alpha}$. Thus, the discussion of Section 2.2.6 shows that our field equations (22) are covariant under the space-time conservation group.

We now consider how our field equations transform under frame transformations. Since a frame transformation leaves the Latin metric unchanged, we easily see that it also leaves the Greek metric unchanged; therefore, it leaves $\sqrt{-g}$ unchanged as well. Thus, under a frame transformation, we have $\sqrt{-g}\, h_i{}^\mu = \sqrt{-g}\, h_{\bar m}{}^\mu x^{\bar m}{}_{,i}$. If we differentiate with respect to x^μ, we obtain

$$(\sqrt{-g}\, h_i{}^\mu)_{,\mu} = (\sqrt{-g}\, h_{\bar m}{}^\mu)_{,\mu} x^{\bar m}{}_{,i} + \sqrt{-g}\, x^{\bar m}{}_{,i,\bar m}$$

From this and equation (13), we see that equation (22) is covariant under the frame conservation group.

3.4. Local Gauge Invariance as an Approximation to Local Frame Invariance

In Section 2.1 we presented an argument which suggests that the $f^i{}_{\mu\nu}$ might somehow collectively describe the Yang–Mills fields of modern gauge theory. We now take the crucial step which allows us to see that this is indeed the case. We define

$$F^i{}_{\mu\nu} = \partial_\mu h^i{}_\nu - \partial_\nu h^i_\mu + C^i{}_{jk} h^j{}_\mu h^k{}_\nu \tag{29}$$

where the C_{ijk} are the real, totally antisymmetric structure constants of a Lie group; see e.g., Glashow and Gell-Mann (1961). We note that $F^i{}_{\mu\nu}$ is the usual field strength for a local Yang–Mills gauge theory, *if $h^i{}_\mu$ is transformed on its Latin indices as a gauge potential*, rather than via a local frame transformation. By using the antisymmetry of C_{ijk} in i and k, we easily verify that $h_i{}^\nu C^i{}_{jk} h^j{}_\mu h^k{}_\nu = 0$. From this and equation (29), we see that equation (27) may be written

$$C_\mu = -h_i{}^\nu F^i{}_{\mu\nu} \tag{30}$$

i.e., that *the form of our field equations is unchanged when the curl $f^i{}_{\mu\nu}$ is replaced by the Yang–Mills field $F^i{}_{\mu\nu}$*. This surprising result provides the clue that leads us to the correct physical interpretation of our theory.

There is no reason to consider transforming $h^i{}_\mu$ as a gauge potential if one recognizes the fundamental role played by the conservation group. The *raison d'être* for this unorthodox way of transforming $h^i{}_\mu$ is that the group of frame diffeomorphisms contains global frame transformations, but contains no local frame transformations. (It is the frame conservation group that contains local frame transformations, as discussed in Section 2.2.7). Thus, when one tries to replace global frame transformations with local frame transformations, analogous to the step taken by Yang and Mills (1954), one fails unless one considers transformations belonging to the conservation group. One can, of course, introduce the familiar local gauge transformations, but our theory is not invariant under these transformations. They are only an approximation to the needed local frame transformations; therefore, they appear in the guise of a broken gauge symmetry. We now illustrate this idea by using the local three-dimensional frame rotation group $O(3)$ as an example. [Of course, it is known that $O(3)$ and $SU(2)$ are homomorphic.] Yang and Mills recognized that a gauge potential has a very cumbersome transformation law, but that one need only consider the infinitesimal transformations. For infinitesimal local $SU(2)$ gauge transformations, their result is

$$h^{\bar{I}}{}_\mu = h^I{}_\mu + e^I{}_{JK} h^J{}_\mu \theta^K + \theta^K{}_{,\mu} \tag{31}$$

where capital Latin indices take the values 1, 2, 3, and e_{IJK} is the usual Levi-Civita symbol.

By contrast, we note that $h^I{}_\mu$ transforms as a vector under a local $O(3)$ frame transformation. Thus, for infinitesimal local $O(3)$ frame transformations

$$h^{\bar{I}}{}_\mu = h^I{}_\mu + e^I{}_{JK} h^J{}_\mu \theta^K \tag{32}$$

For global transformations (i.e., for constant θ^K), equations (31) and (32) are identical. For local transformations, however, the $\theta^K{}_{,\mu}$ term in (31) makes it clear that this equation does not describe a local frame transformation; therefore, the local $SU(2)$ gauge group is only an approximate symmetry in our theory.

We see from equation (31) that when $h^i{}_\mu$ is transformed on its Latin indices as a gauge potential, the metric $g_{\mu\nu} = g_{ij} h^i{}_\mu h^j{}_\nu$ is generally changed. It is eminently reasonable that when a particle is subjected to a gauge transformation which changes its mass, the gravitational field also should change.

3.5. Physical Yang–Mills Field from Ricci Rotation Coefficients

Since $\sqrt{-g}\, h_i{}^\mu$ is a tensor density of weight $+1$, we see that (22) may be written $(\sqrt{-g}\, h_i{}^\mu)_{;\mu} = 0$, where a semicolon denotes the usual covariant differentiation with respect to the Christoffel symbol. Then, since $g_{\mu\nu}$ is covariant constant, we have

$$h_i{}^\mu{}_{;\mu} = 0 \tag{33}$$

Now, the Ricci rotation coefficients (Eisenhart, 1925) are defined by $\gamma_{i\nu\alpha} = h_{i\nu;\alpha}$; and the relation $\gamma_{\mu\nu\alpha} = h^i{}_\mu \gamma_{i\nu\alpha}$ illustrates our general method for converting between Greek and Latin indices. These coefficients are antisymmetric in their first two indices, i.e., $\gamma_{\nu\mu\alpha} = -\gamma_{\mu\nu\alpha}$. By using this property, we easily see that

$$C_\mu = \gamma^\nu{}_{\mu\nu} \tag{34}$$

1. Permutation Group Decomposition of the Ricci Rotation Coefficients. The permutation group of degree three has six group elements. One group element is the identity. The other five group elements are "cycles" such as $(\mu\nu\alpha)$, which has the effect of replacing μ with ν, ν with α, and α with μ. These five group elements are $(\mu\nu)$, $(\nu\alpha)$, $(\alpha\mu)$, $(\mu\alpha\nu)$, and $(\mu\nu\alpha)$. The Ricci rotation coefficients may be decomposed into their totally antisymmetric and mixed symmetry parts. (The totally symmetric part vanishes because the coefficients are antisymmetric in their first two indices.) The totally antisymmetric part of $\gamma_{\mu\nu\alpha}$ is

$$A_{\mu\nu\alpha} = \tfrac{1}{3}(\gamma_{\mu\nu\alpha} + \gamma_{\alpha\mu\nu} + \gamma_{\nu\alpha\mu}) \tag{35}$$

The mixed symmetry part of $\gamma_{\mu\nu\alpha}$ is the quantity

$$M_{\mu\nu\alpha} = \gamma_{\mu\nu\alpha} - A_{\mu\nu\alpha} = \tfrac{1}{3}(2\gamma_{\mu\nu\alpha} - \gamma_{\alpha\mu\nu} - \gamma_{\nu\alpha\mu}) \tag{36}$$

which is antisymmetric in μ and ν. Thus, we have

$$\gamma_{\mu\nu\alpha} = A_{\mu\nu\alpha} + M_{\mu\nu\alpha} \tag{37}$$

and we see from equations (34) and (37) that

$$C_\mu = M^\nu{}_{\mu\nu} \tag{38}$$

It follows from equation (36) that

$$M_{\mu\nu\alpha} + M_{\alpha\mu\nu} + M_{\nu\alpha\mu} = 0 \tag{39}$$

We note that $M_{i\mu\nu}$ may be expressed in terms of $f_{i\mu\nu}$, the curl of $h_{i\mu}$. We have $f_{i\mu\nu} = h_{i\nu,\mu} - h_{i\mu,\nu} = h_{i\nu;\mu} - h_{i\mu;\nu}$, so that $f_{i\mu\nu} = \gamma_{i\nu\mu} - \gamma_{i\mu\nu}$. If we subtract from this the corresponding expressions for $f_{\mu\nu i}$ and $f_{\nu i\mu}$, we see that

$$\gamma_{\mu\nu i} = \tfrac{1}{2}(f_{i\mu\nu} - f_{\mu\nu i} - f_{\nu i\mu}) \tag{40}$$

From equations (36) and (40) we find that

$$M_{\mu\nu i} = \tfrac{1}{3}(2f_{i\mu\nu} - f_{\mu\nu i} - f_{\nu i\mu}) \tag{41}$$

and this may be written

$$M_{\mu\nu i} = \tfrac{1}{3}(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h^n{}_\mu\delta_\nu^\alpha h_i{}^\sigma - h^n{}_\nu h_i{}^\alpha\delta_\mu^\sigma)f_{n\alpha\sigma} \tag{42}$$

Upon using equations (28) and (29) in equation (42), we obtain

$$M_{\mu\nu i} = \tfrac{1}{3}(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h^n{}_\mu\delta_\nu^\alpha h_i{}^\sigma - h^n{}_\nu h_i{}^\alpha\delta_\mu^\sigma)(F_{n\alpha\sigma} - C_{njk}h^j{}_\alpha h^k{}_\sigma) \tag{43}$$

It is easily verified that

$$(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h^n{}_\mu\delta_\nu^\alpha h_i{}^\sigma - h^n{}_\nu h_i{}^\alpha\delta_\mu^\sigma)C_{njk}h^j{}_\alpha h^k{}_\sigma = 0$$

Therefore, equation (42) reduces to

$$M_{\mu\nu i} = \tfrac{1}{3}(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h^n{}_\mu\delta_\nu^\alpha h_i{}^\sigma - h^n{}_\nu h_i{}^\alpha\delta_\mu^\sigma)F_{n\alpha\sigma} \tag{44}$$

From equation (44), we see that in expression (42) for $M_{\mu\nu i}$, the curl $f_{i\mu\nu}$ may simply be replaced by the gauge field $F_{i\mu\nu}$ [just as in expression (27) for C_μ]. We shall see that the quantity $F_{i\mu\nu}$ does not directly describe the physical Yang–Mills field that appears in the stress-energy tensor $T_{\mu\nu}$ of Einstein's equations. It is, however, the fundamental ingredient which is essential for the description of that field. Indeed, $F_{i\alpha\sigma}$ in equation (44) may be viewed as a field with "bare" or massless quanta, which are "clothed" by the factor $\tfrac{1}{3}(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h^n{}_\mu\delta_\nu^\alpha h_i{}^\sigma - h^n{}_\nu h_i{}^\alpha\delta_\mu^\sigma)$, and thus become massive. It is $M_{\mu\nu i}$ that appears in $T_{\mu\nu}$ as a field with massive quanta.

2. *Affine Connection for Quantities with Latin Indices.* We shall see that it is useful to regard the negative of $A^i{}_{j\nu}$ as an affine connection for

"total" covariant differentiation of quantities with Latin indices. We use a stroke | to denote the total covariant derivative. Thus, for the total covariant derivatives of $h^i{}_\mu$ and $h_{i\mu}$, we have

$$h^i{}_{\mu|\nu} = h^i{}_{\mu,\nu} - h^i{}_\alpha \Gamma^\alpha{}_{\mu\nu} - h^j{}_\mu A^i{}_{j\nu} = h^i{}_{\mu;\nu} - h^j{}_\mu A^i{}_{j\nu} = M^i{}_{\mu\nu} \tag{45}$$

and $h_{i\mu|\nu} = h_{j\mu;\nu} + h_{j\mu} A^j{}_{i\nu} = M_{i\mu\nu}$. (For a quantity that has only Greek indices, there is no distinction between "ordinary" and "total" covariant differentiation; e.g., $C_{\mu|\nu} = C_{\mu;\nu}$.)

We note that our total covariant differentiation corresponds to the use of a spin connection with torsion. It is known (Hatfield, 1992) that the imposition of additional symmetries, such as supersymmetry, may require the use of such a spin connection.

It is important to observe from equations (39) and (45) that $M_{\mu\nu i}$ is just the "total curl" of $h_{i\mu}$, i.e.,

$$M_{\mu\nu i} = h_{i\nu|\mu} - h_{i\mu|\nu} \tag{46}$$

This observation encourages us to identify $M_{\mu\nu i}$ tentatively as the "physical" Yang–Mills field. For this identification to be valid, the quantity

$$M_{\mu\nu 0} = \tfrac{1}{3}(2f_{0\mu\nu} - f_{\mu\nu 0} - f_{\nu 0\mu}) \tag{47}$$

must describe the electromagnetic field. The presence of the extra terms $-f_{\mu\nu 0} - f_{\nu 0\mu}$ in equation (47) may cause one to ask how $M_{\mu\nu i}$ can be identified as the physical Yang–Mills field. Our answer is this: The orthodox interpretation of $h^i{}_\mu$ is that it describes an observer frame; and, *if $h^i{}_\mu$ describes a freely falling, nonrotating observer frame, then equation (47) reduces to $M_{\mu\nu 0} = \frac{1}{3}f_{0\mu\nu}$*. This may be seen as follows. The vector field $h^0{}_\mu$ is tangent to, and therefore defines, a timelike congruence of curves. These are the world lines of an observer with velocity $h^0{}_\mu$ carrying a "spatial" frame described by $h^I{}_\mu$, where capital Latin indices take the values $1, \ldots, N-1$. To obtain an $h^i{}_\mu$ that describes a freely falling, nonrotating frame, we choose $h^0{}_\mu$ tangent to a timelike geodesic congruence, and carry $h^I{}_\mu$ along the geodesics by parallel transport [to which Fermi–Walker transport reduces (Synge, 1960) along nonnull geodesics]. Thus, the condition for freely falling, nonrotating frames is $h_{i\nu;\alpha} h_0{}^\alpha = 0$. In terms of the Ricci rotation coefficients, the condition is $\gamma_{\mu\nu 0} = 0$. From this and (36), we see that for an $h^i{}_\mu$ which describes a freely falling, nonrotating observer frame,

$$M_{\mu\nu 0} = \tfrac{1}{3}(\gamma_{0\nu\mu} - \gamma_{0\mu\nu}) = \tfrac{1}{3}(h_{0\nu;\mu} - h_{0\mu;\nu}) = \tfrac{1}{3}(h_{0\nu,\mu} - h_{0\mu,\nu}) = \tfrac{1}{3}f_{0\mu\nu}$$

4. TOTAL EINSTEIN EQUATIONS

We now present compelling evidence that $M_{\mu\nu}{}^i$ describes the physical Yang–Mills field. We define a "total" Riemann tensor

$$\mathfrak{R}^\alpha{}_{\beta\mu\nu} = h_i{}^\alpha (h^i{}_{\beta|\mu|\nu} - h^i{}_{\beta|\nu|\mu}) \tag{48}$$

which is the total analog of the usual Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$. We define a total Ricci tensor by $\mathfrak{R}_{\mu\nu} = \mathfrak{R}^{\alpha}{}_{\mu\alpha\nu}$, and a total Ricci scalar by $\mathfrak{R} = \mathfrak{R}^{\alpha}{}_{\alpha}$. By using

$$h_i{}^{\alpha}h^i{}_{\beta|\mu|\nu} = (h_i{}^{\alpha}h^i{}_{\beta|\mu})_{|\nu} - h_i{}^{\alpha}{}_{|\nu}h^i{}_{\beta|\mu} = M^{\alpha}{}_{\beta\mu|\nu} + M^{\alpha}{}_{\sigma\nu}M^{\sigma}{}_{\beta\mu}$$

we easily find from (48) that

$$\mathfrak{R}^{\alpha}{}_{\beta\mu\nu} = M^{\alpha}{}_{\beta\mu|\nu} - M^{\alpha}{}_{\beta\nu|\mu} + M^{\alpha}{}_{\sigma\nu}M^{\sigma}{}_{\beta\mu} - M^{\alpha}{}_{\sigma\mu}M^{\sigma}{}_{\beta\nu} \tag{49}$$

From equations (38) and (49), we find that

$$\mathfrak{R}_{\mu\nu} = C_{\mu|\nu} - C_{\alpha}M^{\alpha}{}_{\mu\nu} - M^{\alpha}{}_{\mu\nu|\alpha} + M^{\sigma}{}_{\alpha\nu}M^{\alpha}{}_{\mu\sigma} \tag{50}$$

so that

$$\mathfrak{R} = 2C^{\alpha}{}_{|\alpha} + C^{\alpha}C_{\alpha} + M^{\sigma\alpha\beta}M_{\beta\sigma\alpha} \tag{51}$$

4.1. An Identity for the Total Einstein Tensor

We define a total Einstein tensor by $\mathfrak{G}_{\mu\nu} = \mathfrak{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathfrak{R}$, and we find from equations (50) and (51) that

$$\begin{aligned}\mathfrak{G}_{\mu\nu} = {} & C_{\mu|\nu} - C_{\alpha}M^{\alpha}{}_{\mu\nu} - g_{\mu\nu}C^{\alpha}{}_{|\alpha} - \tfrac{1}{2}g_{\mu\nu}C^{\alpha}C_{\alpha} \\ & - M^{\alpha}{}_{\mu\nu|\alpha} + M^{\alpha}{}_{\sigma\nu}M^{\sigma}{}_{\mu\alpha} - \tfrac{1}{2}g_{\mu\nu}M^{\sigma\alpha\beta}M_{\beta\sigma\alpha}\end{aligned} \tag{52}$$

Next, we notice that

$$\begin{aligned}-M^{\alpha}{}_{\mu\nu|\alpha} &= M_{\mu}{}^{\alpha}{}_{\nu|\alpha} = (M_{\mu}{}^{\alpha}{}_i h^i{}_{\nu})_{|\alpha} = M_{\mu}{}^{\alpha}{}_{i|\alpha}h^i{}_{\nu} + M_{\mu}{}^{\alpha}{}_i h^i{}_{\nu|\alpha} \\ &= M_{\mu}{}^{\alpha}{}_{i|\alpha}h^i{}_{\nu} + M_{\mu}{}^{\alpha}{}_i M^i{}_{\nu\alpha} = M_{\mu}{}^{\alpha}{}_{i|\alpha}h^i{}_{\nu} + M_{\mu}{}^{\alpha}{}_{\sigma}M^{\sigma}{}_{\nu\alpha} \\ &= M_{\mu}{}^{\alpha}{}_{i|\alpha}h^i{}_{\nu} + M_{\mu}{}^{\sigma}{}_{\alpha}M^{\alpha}{}_{\nu\sigma}\end{aligned}$$

From this and equation (52) we have

$$\begin{aligned}\mathfrak{G}_{\mu\nu} = {} & C_{\mu|\nu} - C_{\alpha}M^{\alpha}{}_{\mu\nu} - g_{\mu\nu}C^{\alpha}{}_{|\alpha} - \tfrac{1}{2}g_{\mu\nu}C^{\alpha}C_{\alpha} \\ & + M_{\mu}{}^{\alpha}{}_{i|\alpha}h^i{}_{\nu} + M_{\mu}{}^{\sigma}{}_{\alpha}M^{\alpha}{}_{\nu\sigma} + M^{\sigma}{}_{\mu\alpha}M^{\alpha}{}_{\sigma\nu} - \tfrac{1}{2}g_{\mu\nu}M^{\alpha\sigma\beta}M_{\beta\sigma\alpha}\end{aligned} \tag{53}$$

Now,

$$\begin{aligned}M_{\mu}{}^{\sigma}{}_{\alpha}M^{\alpha}{}_{\nu\sigma} + M^{\sigma}{}_{\mu\alpha}M^{\alpha}{}_{\sigma\nu} &= M^{\sigma}{}_{\mu\alpha}M_{\nu}{}^{\alpha}{}_{\sigma} + M^{\sigma}{}_{\mu\alpha}M^{\alpha}{}_{\sigma\nu} \\ &= M^{\sigma}{}_{\mu\alpha}(M_{\nu}{}^{\alpha}{}_{\sigma} + M^{\alpha}{}_{\sigma\nu})\end{aligned}$$

so, by using (39), we have

$$M_{\mu}{}^{\sigma}{}_{\alpha}M^{\alpha}{}_{\nu\sigma} + M^{\sigma}{}_{\mu\alpha}M^{\alpha}{}_{\sigma\nu} = -M^{\sigma}{}_{\mu\alpha}M_{\sigma\nu}{}^{\alpha} = -M^{\alpha}{}_{\mu i}M_{\alpha\nu}{}^{i} \tag{54}$$

Similarly,

$$\begin{aligned} M^{\sigma\alpha\beta}M_{\beta\sigma\alpha} &= \tfrac{1}{2}(M^{\sigma\alpha\beta}M_{\beta\sigma\alpha} + M^{\sigma\alpha\beta}M_{\beta\sigma\alpha}) \\ &= \tfrac{1}{2}(M^{\sigma\alpha\beta}M_{\beta\sigma\alpha} + M^{\beta\alpha\sigma}M_{\sigma\beta\alpha}) \\ &= \tfrac{1}{2}(M^{\sigma\alpha\beta}M_{\beta\sigma\alpha} + M^{\alpha\beta\sigma}M_{\beta\sigma\alpha}) \\ &= \tfrac{1}{2}(M^{\sigma\alpha\beta} + M^{\alpha\beta\sigma})M_{\beta\sigma\alpha} \end{aligned}$$

and, by using Eq. (39), we have

$$M^{\sigma\alpha\beta}M_{\beta\sigma\alpha} = -\tfrac{1}{2}M^{\beta\sigma\alpha}M_{\beta\sigma\alpha} = -\tfrac{1}{2}M^{\beta\sigma i}M_{\beta\sigma i} \tag{55}$$

From equations (53)–(55), we find that an identity for the total Einstein tensor is

$$\begin{aligned} \mathfrak{G}_{\mu\nu} = {} & C_{\mu|\nu} - C_\alpha M^\alpha{}_{\mu\nu} - g_{\mu\nu}C^\alpha{}_{|\alpha} - \tfrac{1}{2}g_{\mu\nu}C^\alpha C_\alpha \\ & + J_{\mu i}h^i{}_\nu - (M^\alpha{}_{\mu i}M_{\alpha\nu}{}^i - \tfrac{1}{4}g_{\mu\nu}M^{\alpha\beta i}M_{\alpha\beta i}) \end{aligned} \tag{56}$$

where $J_{\mu i} = M_\mu{}^\alpha{}_{i|\alpha}$ is the total Yang–Mills current. It is not generally conserved.

4.2. Total Einstein Equations

Our field equations (26) just state that $C_\mu = 0$. Thus, we see from (56) that the field equations imply the validity of the total Einstein equations

$$\mathfrak{G}_{\mu\nu} = J_{\mu i}h^i{}_\nu - (M^\alpha{}_{\mu i}M_{\alpha\nu}{}^i - \tfrac{1}{4}g_{\mu\nu}M^{\alpha\beta i}M_{\alpha\beta i}) \tag{57}$$

By contrast with the conventional Einstein tensor $G_{\mu\nu}$, our total Einstein tensor $\mathfrak{G}_{\mu\nu}$ is nonsymmetric. We denote its symmetric part by $\mathfrak{G}_{\underline{\mu\nu}}$. The symmetric part of equation (57) is

$$\mathfrak{G}_{\underline{\mu\nu}} = \tfrac{1}{2}(J_{\mu i}h^i{}_\nu + J_{\nu i}h^i{}_\mu) - (M^\alpha{}_{\mu i}M_{\alpha\nu}{}^i - \tfrac{1}{4}g_{\mu\nu}M^{\alpha\beta i}M_{\alpha\beta i}) \tag{58}$$

The right side of equation (58) is just what one expects for the stress-energy tensor of a non-Abelian Yang–Mills field and associated currents.

5. SOLUTIONS WITH PATH-INDEPENDENT $h^i{}_\mu$

5.1. The General Construction

In Section 2.2.5 we stated that there exists a large class of solutions to our field equations for which $h^i{}_\mu$ is path independent. We now exhibit this class.

Let $H_i^{\mu\nu}$ be N antisymmetric tensor densities of weight $+1$ under space-time (Greek) coordinate transformations. The only conditions on the $H_i^{\mu\nu}$ are that:

1. They be path-independent functions, i.e., that $H_i^{\mu\nu} = H_i^{\mu\nu}(x)$.
2. The vector densities of weight $+1$ defined by $H_i^{\mu} = H_i^{\mu\nu}{}_{,\nu}$ be linearly independent.

From Condition 1, it follows that $[\partial_\alpha, \partial_\beta]H_i^{\mu\nu} = 0$. From Condition 2, it follows that H, the determinant of H_i^{μ}, is nonzero. This determinant is

$$H = \frac{1}{N!}\, e_{\alpha\sigma\ldots\mu} H_i^{\alpha} H_j^{\sigma} \cdots H_m^{\mu} e^{ij\cdots m} \tag{59}$$

It is clear from equation (59) that H is a scalar density of weight $N - 1$. Thus, $H^{1/(1-N)}$ is a scalar density of weight -1, so that $H^{1/(1-N)}H_i^{\mu}$ is a vector, i.e., it has weight zero. We define h_i^{μ} by

$$h_i^{\mu} = H^{1/(1-N)} H_i^{\mu} \tag{60}$$

and, of course, $h^i{}_\mu$ is defined by $h^i{}_\mu h_i^{\nu} = \delta^\nu_\mu$, while $g_{\mu\nu} = g_{ij} h^i{}_\mu h^j{}_\nu$, as discussed previously. Thus, we see that

$$\begin{aligned}\sqrt{-g} = \operatorname{Det} h^i{}_\mu = (\operatorname{Det} h_i^{\mu})^{-1} &= \{\operatorname{Det}[H^{1/(1-N)} H_i^{\mu}]\}^{-1} \\ &= [H^{N/(1-N)} H]^{-1} = H^{1/(N-1)}\end{aligned}$$

From this and equation (60), we see that

$$\sqrt{-g}\, h_i^{\mu} = H_i^{\mu} \tag{61}$$

By using equation (61), we find that $(\sqrt{-g}\, h_i^{\mu})_{,\mu} = 0$, i.e., that our field equations are satisfied. This is easily verified as follows:

$$\begin{aligned}(\sqrt{-g}\, h_i^{\mu})_{,\mu} &= H_i^{\mu}{}_{,\mu} = H_i^{\mu\nu}{}_{,\nu,\mu} \\ &= \tfrac{1}{2}(H_i^{\mu\nu}{}_{,\nu,\mu} + H_i^{\mu\nu}{}_{,\nu,\mu}) = \tfrac{1}{2}(H_i^{\mu\nu}{}_{,\nu,\mu} + H_i^{\nu\mu}{}_{,\mu,\nu}) \\ &= \tfrac{1}{2}(H_i^{\mu\nu}{}_{,\nu,\mu} - H_i^{\mu\nu}{}_{,\mu,\nu}) = \tfrac{1}{2}[\partial_\mu, \partial_\nu] H_i^{\mu\nu} = 0\end{aligned}$$

5.2. Direct Product of Compact n-Dimensional Space and Flat Four-Dimensional Space-Time

A slight modification of the construction described in Section 5.1 shows that the field equations admit solutions which describe the direct product of a compact n-dimensional space and flat four-dimensional space-time. Let $H_i^{\mu} = \delta_i^{\mu}$ if either i or μ is in 0, 1, 2, 3. Let $H_i^{\mu} = H_i^{\mu\nu}{}_{,\nu}$ if both i and μ are in $4, 5, \ldots, N-1$, and let the antisymmetric $H_i^{\mu\nu}$ be path-independent functions $H_i^{\mu\nu}(x)$ that do not depend upon x^α for $\alpha = 0, 1, 2, 3$. Let H, the determinant of H_i^{μ}, be constant. Define $h_i^{\mu} = H_i^{\mu}$. With this construction, it is easily verified that the field equations are satisfied, and that $g_{\mu\nu}$ is in block form. More specifically, it is seen that $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ if both μ and ν are in 0, 1, 2, 3; and that $g_{\mu\nu} = 0$ if either μ or ν is in 0, 1, 2, 3 and the other is in 4, 5, . . . , $N - 1$. If both μ and ν are in 4, 5, . . . , $N - 1$, we see that that $g_{\mu\nu}$ is largely unrestricted because the condition $H = \text{const}$ imposes only one restriction on the $\frac{1}{2}n^2(n - 1)$ functions $H_i^{\mu\nu}$ that are present for $n > 1$.

6. AN ALTERNATE THEORY

We previously (Pandres, 1981) considered the variational principle

$$\delta \int C^{\mu} C_{\mu} \sqrt{-g}\, d^N x = 0 \tag{62}$$

where h_i^{μ} is varied. We note that $\sqrt{-g}$ equals h, the determinant of $h^i{}_{\mu}$; and that $g^{\mu\nu} C_{\mu} C_{\nu} = g^{ij} C_i C_j$, where $C_i = C_{\mu} h_i^{\mu}$. Hence, equation (62) may be written

$$\delta \int g^{ij} C_i C_j h\, d^N x = 0 \tag{63}$$

The variational calculation using equation (63) is a bit less difficult than that using equation (62). From equations (27) and (28), we easily see that

$$C_i = -h^j{}_{\mu}(h_i{}^{\mu}{}_{,j} - h_j{}^{\mu}{}_{,i}) \tag{64}$$

We find from equation (63) that

$$\int (2C^i \delta C_i h - C^i C_i h h^k{}_{\alpha} \delta h_k{}^{\alpha})\, d^N x = 0 \tag{65}$$

where we have used the relation $\delta h = h h_k{}^{\alpha} \delta h^k{}_{\alpha} = -h h^k{}_{\alpha} \delta h_k{}^{\alpha}$. We note that

$$\begin{aligned} h^{-1}(h h_i{}^{\nu})_{,\nu} &= h^{-1}(h_{,\nu} h_i{}^{\nu} + h h_i{}^{\nu}{}_{,\nu}) = h^{-1}(h_{,i} + h h_i{}^{\nu}{}_{,\nu}) \\ &= h^{-1}(h h_j{}^{\nu} h^j{}_{\nu,i} + h h_i{}^{\nu}{}_{,\nu}) = -h_j{}^{\nu}{}_{,i} h^j{}_{\nu} + h_i{}^{\nu}{}_{,j} h^j{}_{\nu} \end{aligned}$$

From this and equation (64), we see that

$$C_i = -h^{-1}(h h_i{}^{\nu})_{,\nu} \tag{66}$$

From equation (66), we have

$$\begin{aligned} \delta C_i &= (h h_i{}^{\alpha})_{,\alpha} h^{-2} \delta h - h^{-1} \delta[(h h_i{}^{\alpha})_{,\alpha}] \\ &= (h h_i{}^{\alpha})_{,\alpha} h^{-2} h h_j{}^{\beta} \delta h^j{}_{\beta} - h^{-1}[\delta(h h_i{}^{\alpha})]_{,\alpha} \\ &= -h^{-1}(h h_i{}^{\alpha})_{,\alpha} h^j{}_{\beta} \delta h_j{}^{\beta} - h^{-1}[\delta(h h_i{}^{\alpha})]_{,\alpha} \\ &= C_i h^j{}_{\beta} \delta h_j{}^{\beta} - h^{-1}[\delta(h h_i{}^{\alpha})]_{,\alpha} \end{aligned}$$

so that

$$\delta C_i = C_i h^j{}_\beta \delta h_j{}^\beta - h^{-1}[\delta(hh_i{}^\alpha)]_{,\alpha} \tag{67}$$

Upon using equation (67) in (65), we obtain

$$\int C^k C_k h h^i{}_\alpha \delta h_i{}^\alpha \, d^N x - 2 \int C^i [\delta(hh_i{}^\alpha)]_{,\alpha} \, d^N x = 0 \tag{68}$$

and integration by parts gives

$$\int h(C^i{}_{,\alpha} - h^i{}_\alpha C^k{}_{,k} + \tfrac{1}{2} h^i{}_\alpha C^k C_k) \delta h_i{}^\alpha \, d^N x - \int [C^i \delta(hh_i{}^\alpha)]_{,\alpha} \, d^N x = 0 \tag{69}$$

By using Gauss' theorem, we may write the second integral of equation (69) as an integral over the boundary of the region of integration of the arithmetic space X. We discard this boundary integral by demanding that $C^i\delta(hh_i{}^\alpha)$ shall vanish on the boundary. Thus, we obtain

$$\int h(C^i{}_{,\alpha} - h^i{}_\alpha C^k{}_{,k} + \tfrac{1}{2} h^i{}_\alpha C^k C_k) \, \delta h_i{}^\alpha \, d^N x = 0 \tag{70}$$

and, by demanding that $\delta h_i{}^\alpha$ be arbitrary in the interior of the (arbitrary) region of integration in X, we obtain the field equations

$$C^i{}_{,\alpha} - h^i{}_\alpha C^k{}_{,k} + \tfrac{1}{2} h^i{}_\alpha C^k C_k = 0 \tag{71}$$

Upon multiplying equation (71) by $h_j{}^\alpha$, we obtain

$$C^i{}_{,j} - \delta^i_j C^k{}_{,k} + \tfrac{1}{2} \delta^i_j C^k C_k = 0 \tag{72}$$

We note that

$$C^\alpha{}_{;\sigma} = (C^k h_k{}^\alpha)_{;\sigma} = C^k{}_{,\sigma} h_k{}^\alpha + C^k h_k{}^\alpha{}_{;\sigma} = C^k{}_{,\sigma} h_k{}^\alpha + C^k \gamma_k{}^\alpha{}_\sigma$$

Thus, we have $C^k{}_{,\sigma} h_k{}^\alpha = C^\alpha{}_{;\sigma} - C^\rho \gamma_\rho{}^\alpha{}_\sigma = C^\alpha{}_{;\sigma} + C^\rho \gamma^\alpha{}_{\rho\sigma}$. Upon multiplying by $h^i{}_\alpha h_j{}^\sigma$, we obtain

$$C^i{}_{,j} = h^i{}_\alpha h_j{}^\sigma (C^\alpha{}_{;\sigma} + C^\rho \gamma^\alpha{}_{\rho\sigma}) \tag{73}$$

and

$$C^k{}_{,k} = C^\alpha{}_{;\alpha} + C^\alpha C_\alpha \tag{74}$$

If we use equations (73) and (74) in equation (72), we get

$$h^i{}_\alpha h_j{}^\sigma (C^\alpha{}_{;\sigma} + C^\rho \gamma^\alpha{}_{\rho\sigma}) - \delta^i_j C^\alpha{}_{;\alpha} - \tfrac{1}{2} \delta^i_j C^\alpha C_\alpha = 0 \tag{75}$$

and, upon multiplying equation (75) by $h_{i\mu} h^j{}_\nu$, we obtain

$$C_{\mu;\nu} - C_\alpha \gamma^\alpha{}_{\mu\nu} - g_{\mu\nu} C^\alpha{}_{;\alpha} - \tfrac{1}{2} g_{\mu\nu} C^\alpha C_\alpha = 0 \tag{76}$$

which are the field equations for our alternate theory.

As we noted following equation (45), for a quantity that has only Greek indices, there is no distinction between "ordinary" and "total" covariant differentiation. From this and equation (37) we see that equation (56), our identity for the total Einstein tensor, may be written

$$\begin{aligned} \mathfrak{G}_{\mu\nu} = {} & C_{\mu;\nu} - C_\alpha \gamma^\alpha{}_{\mu\nu} - g_{\mu\nu} C^\alpha{}_{;\alpha} - \tfrac{1}{2} g_{\mu\nu} C^\alpha C_\alpha \\ & + J_{\mu i} h^i{}_\nu - (M^\alpha{}_{\mu i} M_{\alpha\nu}{}^i - \tfrac{1}{4} g_{\mu\nu} M^{\alpha\beta i} M_{\alpha\beta i}) + C_\alpha A^\alpha{}_{\mu\nu} \end{aligned} \tag{77}$$

Now, equation (76) just states that the first line on the right side of equation (77) vanishes. Thus, equation (76) may be written

$$\mathfrak{G}_{\mu\nu} = J_{\mu i} h^i{}_\nu - (M^\alpha{}_{\mu i} M_{\alpha\nu}{}^i - \tfrac{1}{4} g_{\mu\nu} M^{\alpha\beta i} M_{\alpha\beta i}) + C_\alpha A^\alpha{}_{\mu\nu} \tag{78}$$

We note that equation (78) differs from equation (57) only by the term $C_\alpha A^\alpha{}_{\mu\nu}$, which is antisymmetric in μ and ν. Thus, the symmetric part of equation (78) is identical in form to equation (58), but with $h^i{}_\mu$ rather than x^i as fundamental variables.

We conclude this paper by considering the relative merits of this alternate theory and the theory developed in Sections 1–5.

6.1. Advantages of the Theory That Flows from $\delta \int \sqrt{-g}\, d^N x = 0$

First, the variational principle $\delta \int \sqrt{-g}\, d^N x = 0$ is simpler than $\delta \int C^\mu C_\mu \sqrt{-g}\, d^N x = 0$. That it can be interpreted as a "principle of stationary volume" is clear, since the Lagrangian density $\sqrt{-g}$ equals the (Jacobian) determinant of $x^i{}_{,\mu}$. Second, $\sqrt{-g}$ involves only $g_{\mu\nu}$, while $C^\mu C_\mu \sqrt{-g}$ involves both $h^i{}_\mu$ and its derivatives. Third, if one uses $\delta \int C^\mu C_\mu \sqrt{-g}\, d^N x = 0$, one gets field equations that are covariant under the conservation group because one has put conservation group invariance into the Lagrangian by hand. By contrast, conservation group covariance emerges in a natural way from $\delta \int \sqrt{-g}\, d^N x = 0$. Fourth, and perhaps most compelling, $\sqrt{-g}$ is a scalar density of weight $+1$ under *all* path-dependent coordinate transformations, while $C^\mu C_\mu \sqrt{-g}$ is a scalar density of weight $+1$ only under conservative coordinate transformations. We again recall the prophecy of Dirac (1930) that "Further progress lies in the direction of making our equations invariant under wider and still wider transformations." This suggests that the quantum theory obtained via the path integral formalism using the Lagrangian density $\sqrt{-g}$ will be superior to that using the Lagrangian density $C^\mu C_\mu \sqrt{-g}$. Fifth, one can hope that a theory which gives a true description of nature is unique, i.e., that it is in some sense the "only possible theory." There is an intrinsic lack of uniqueness in our alternate theory. This lack of uniqueness arises because if the $h^i{}_\mu$ are varied, then instead of equation (62) one can use the variational principle

$$\delta \int (C^\mu C_\mu + \Lambda)\sqrt{-g}\, d^N x = 0 \tag{79}$$

where Λ is a constant. When one varies $h^i{}_\mu$, one obtains the field equations

$$C_{\mu;\nu} - C_\alpha \gamma^\alpha{}_{\mu\nu} - g_{\mu\nu} C^\alpha{}_{;\alpha} - \tfrac{1}{2} g_{\mu\nu} C^\alpha C_\alpha = \tfrac{1}{2}\Lambda g_{\mu\nu} \tag{80}$$

By using the identity (56), we see that equation (80) leads to

$$\begin{aligned}\mathfrak{G}_{\underline{\mu\nu}} = \tfrac{1}{2}(J_{\mu i} h^i{}_\nu + J_{\nu i} h^i{}_\mu) - (M^\alpha{}_{\mu i} M_{\alpha\nu}{}^i - \tfrac{1}{4} g_{\mu\nu} M^{\alpha\beta i} M_{\alpha\beta i}) \\ + \tfrac{1}{2}\Lambda g_{\mu\nu}\end{aligned} \tag{81}$$

Equation (81) differs from equation (58) through the inclusion of the "cosmological constant" term $\frac{1}{2}\Lambda g_{\mu\nu}$. By contrast, the field equations that flow from $\delta\int(1 + \Lambda)\sqrt{-g}\, d^N x = 0$ are identical with those that flow from $\delta\int\sqrt{-g}\, d^N x = 0$.

6.2. Advantages of the Theory That Flows from $\delta \int C^\mu C_\mu \sqrt{-g}\, d^N x = 0$

First, from the analysis of Section 3 we see that the field equations which flow from $\delta \int \sqrt{-g}\, d^N x = 0$ state that $C_i = 0$. If the requirement $C_i = 0$ should prove too restrictive to meet the test of experiment, one can turn to the theory which flows from $\delta \int C^\mu C_\mu \sqrt{-g}\, d^N x = 0$. We see by inspection of equation (72) that a sufficient condition for a solution to the field equations for this theory is that C_i be constant and that $C^i C_i = 0$. We have shown (Pandres, 1984a) that this condition is also necessary.

Second, we note that from equations (51) and (55) there follows the identity

$$C^\alpha C_\alpha = \mathfrak{R} + \tfrac{1}{2} M^{\alpha\beta i} M_{\alpha\beta i} - 2C^\alpha{}_{;\alpha} \tag{82}$$

This identity shows that the Lagrangian $C^\alpha C_\alpha$ may be viewed as the sum of the total Lagrangians for the gravitational and Yang–Mills fields, plus a divergence term which contributes nothing to the field equations.

Finally, we recall from Section 3 that if $C_\mu = 0$, then there exists a conservative coordinate transformation from x^α to a special $x^{\bar\alpha}$ coordinate system in which $h^i{}_{\bar\mu}$ is constant. Thus, all forces can be viewed as "generalized Coriolis forces." This is consistent with Occams razor—*entia non multiplicanda praeter necessitatem*—but it violates the general relativistic principle that no preferred coordinate system should exist.

ACKNOWLEDGMENTS

It is a pleasure to thank Prof. Martin Ahrens, Prof. David Finkelstein, Prof. Edward L. Green, Dr. Bruce McFarland, Dr. H. G. Loos, Prof. Louis Witten, and Prof. J. D. Zund for stimulating discussions and helpful remarks.

REFERENCES

Appelquist, T., Chodos, A., and Freund, P. G. O. (1987). *Modern Kaluza–Klein Theories,* Addison-Wesley, Redwood City, California.

Davies, P. C. W., and Brown, J. R., eds. (1988). *Superstrings: A Theory of Everything?* Cambridge University Press, Cambridge.

Dirac, P. A. M. (1930). *The Principles of Quantum Mechanics,* 1st ed., Cambridge University Press, Cambridge, Preface.

Dirac, P. A. M. (1978). *Directions in Physics,* Wiley, New York, p. 41.

Eddington, A. (1924). *The Mathematical Theory of Relativity,* Cambridge University Press, Cambridge, p. 225.

Einstein, A. (1949). In *Albert Einstein: Philosopher-Scientist,* P. A. Schilpp, ed., Harper & Brothers, New York, Vol. I, p. 89.

Eisenhart, L. P. (1925). *Riemannian Geometry,* Princeton University Press, Princeton, New Jersey, p. 97.

Finkelstein, D. (1981). Private communication.

Glashow, S. L., and Gell-Mann, M. (1961). *Annals of Physics,* **15**, 437–460.

Green, E. L. (1991). Private communication.

Hatfield, B. (1992). *Quantum Field Theory of Point Particles and Strings,* Addison-Wesley, Redwood City, California, p. 692.

Mandelstam, S. (1962). *Annals of Physics,* **19**, 1–24.

Pandres, D., Jr. (1962). *Journal of Mathematical Physics,* **3**, 602–607.

Pandres, D., Jr. (1973). *Lettere el Nuovo Cimento,* **8**, 595–599.

Pandres, D., Jr. (1977). *Foundations of Physics,* **7**, 421–430.

Pandres, D., Jr. (1981). *Physical Review D,* **24**, 1499–1508.

Pandres, D., Jr. (1984a). *Physical Review D,* **30**, 317–324.

Pandres, D., Jr. (1984b). *International Journal of Theoretical Physics,* **23**, 839–842.

Schrödinger, E. (1950). *Space-Time Structure,* Cambridge University Press, Cambridge, p. 97.

Synge, J. L. (1960). *Relativity: The General Theory,* North-Holland, Amsterdam, p. 14.

Witten, E. (1988). In *Superstrings: A Theory of Everything?,* P. C. W. Davies and J. Brown, eds., Cambridge University Press, Cambridge, p. 90.

Yang, C. N., and Mills, R. L. (1954). *Physical Review,* **96**, 191–195.